

# General Inverses in 2-D, 3-D, applications inspired by Technology

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## Abstract

*We first generalize the concept of inverse images with respect to the line  $y = x$  to the inverse images with respect to  $y = mx + b$ , and we extend the idea to three dimensions. Next, we find the reflection of a point on curve  $C_1$  with respect to a moving point  $P$  on  $C_2$ . We find the locus of the point  $P$ , which links to the concept of orthotomic and caustic curves. The concepts can be extended to the corresponding ones in 3-D. It is interesting to see how technological tools have prompted us to relate Mathematics to similar concepts used in Physics.*

## 1 Introduction

With the aid of [ClassPad] we find the inverse of a curve with respect to a general line of the form  $y = mx + b$ . If the software can simulate the appearance of the reflection curve (as can be seen from section 1.1), then its equation must exist, and if discovered, can be extended to a corresponding result in 3-D. The derivation is explained in Section 1.2. We next find the reflections of a point  $P$  on a curve  $C_1$  with respect to all tangents of another curve  $C_2$  and investigate its locus. It turns out the locus is the orthotomic curve of  $C_2$  relative to  $P$  and the evolute of the orthotomic curve is called the caustic. We describe how we extend the 2-D findings described in section 2 regarding orthotomic and caustic curves to the corresponding ones in 3-D in section 3.

### 1.1 Motivations

We want to find the reflection of  $y = \sqrt{x} + 1$  with respect to  $y = x + 2$ . We refer to this as the general inverse for  $y = f(x)$  with respect to  $y = mx + b$ . We explore this finding by using [ClassPad].

1. First sketch  $y = \sqrt{x} + 1$  and  $y = x + 2$ .

- Pick a point  $A$  on  $y = \sqrt{x} + 1$  and construct the corresponding point  $A'$  that is symmetric to  $A$  with respect to  $y = x + 2$ .
- The trace of  $A'$  is shown in Figure 1 and we would like to find the equation for the trace of  $A'$ .

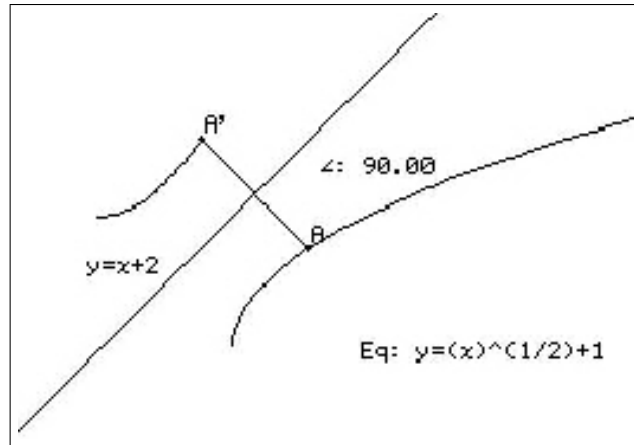


Figure 1. The reflection of  $y = \sqrt{x} + 1$  with respect to  $y = x + 2$

- We summarize above and motivation of deriving some results in this paper in this video clip, see [8]. The corresponding ClassPad eActivity can be found in [9].

**Solution:**

- Assume the general case for finding the inverse of  $[x(t), y(t)]$  with respect to  $y = mx + b$ .
- Set  $\theta = \tan^{-1} m$ .
- We denote the reflection of  $[x(t), y(t)]$  with respect to  $y = mx + b$  by  $[p(t), q(t)]$ .
- We shall derive the following later in section 1.3.

$$\begin{bmatrix} p(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) - b \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix}. \quad (1)$$

First, Let us refer to some examples.

**Example 1** Find the reflection of  $\begin{bmatrix} 2 \cos t - \cos(2t) \\ 2 \sin t - \sin(2t) \end{bmatrix}$  with respect to  $y = 2x + 1$ .

We set  $\theta = \tan^{-1} 2$  and  $b = 1$  in (1) (with ClassPad) to obtain  $\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$  to be

$$\begin{bmatrix} -2 \cos(t + 2 \tan^{-1}(\frac{1}{2})) + \cos(2t + 2 \tan^{-1}(\frac{1}{2})) - \frac{4}{5} \\ \frac{8 \cos t + 6 \sin t - 4 \cos 2t - 3 \sin 2t + 2}{5} \end{bmatrix}. \quad (2)$$

We plot  $\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$  (thick) together with  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$  (thin) below:

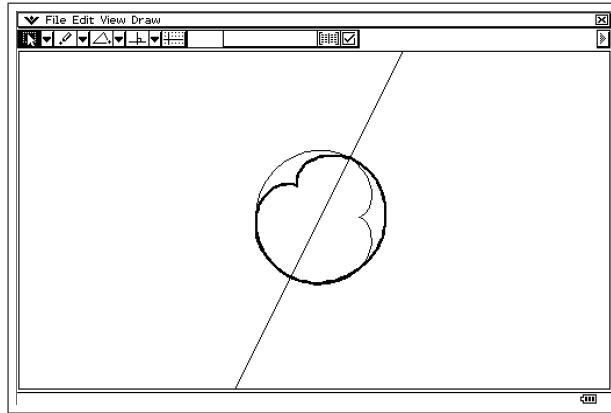


Figure 2. Another reflection.

**Example 2** Find the reflection of the Hypocycloid  $\begin{bmatrix} (a - b) \cos t + b \cos((\frac{a}{b} - 1)t) \\ (a - b) \sin t + b \sin((\frac{a}{b} - 1)t) \end{bmatrix}$  (when  $a = 3$  and  $b = 1$ ) with respect to  $y = 2x + 1$ .

We set  $\theta = \tan^{-1} 2$  and we obtain  $\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$  to be

$$\begin{bmatrix} -2 \cos(t + 2 \tan^{-1}(\frac{1}{2})) - \cos(2t - 2 \tan^{-1}(\frac{1}{2})) - \frac{4}{5} \\ \frac{8 \cos t + 6 \sin t + 4 \cos 2t - 3 \sin 2t + 2}{5} \end{bmatrix}, \quad (3)$$

we plot  $\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$  (thick) together with  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$  (thin) below:

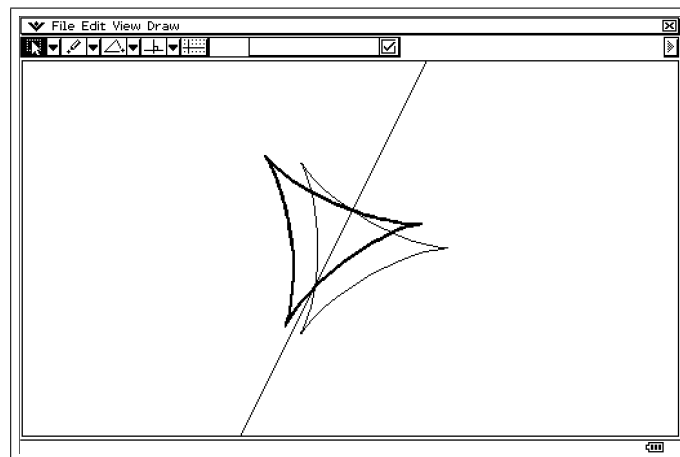


Figure 3. A reflection of Hypocycloid with respect to  $y = 2x + 1$ .

## 1.2 General inverse with respect to a plane

We begin by finding the reflection (denoted by  $[p(t), q(t), r(t)]$ ) of a parametric curve  $[x(t), y(t), z(t)]$  with respect to a plane  $P$  passing through the origin  $ax + by + cz = 0$ . The normal vector of the plane  $P$  is denoted by  $\mathbf{n} = (a, b, c)$ .

We first split the vector  $\mathbf{v}$  (from the origin to  $(x(t), y(t), z(t))$ ) into its components which are normal to the plane (denoted  $\mathbf{v}_{\perp P}$ ) and parallel to the plane (denoted  $\mathbf{v}_{\parallel P}$ ); in other words,  $\mathbf{v} = \mathbf{v}_{\perp P} + \mathbf{v}_{\parallel P}$ . We note that  $\mathbf{v}_{\perp P}$  is the orthogonal projection of  $\mathbf{v}$  on the normal vector  $\mathbf{n} = (a, b, c)$ . Then the reflection of  $[x(t), y(t), z(t)]$  is

$$\begin{aligned} \begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix} &= -\mathbf{v}_{\perp P} + \mathbf{v}_{\parallel P} = -\mathbf{v}_{\perp P} + (\mathbf{v} - \mathbf{v}_{\perp P}) = \mathbf{v} - 2\mathbf{v}_{\perp P} \\ &= \mathbf{v} - 2 \frac{\mathbf{v} \cdot \mathbf{n}}{\|\mathbf{n}\|^2} \mathbf{n} = \frac{\mathbf{v} \|\mathbf{n}\|^2 - 2(\mathbf{v} \cdot \mathbf{n})\mathbf{n}}{\|\mathbf{n}\|^2} \\ &= \frac{1}{a^2 + b^2 + c^2} \begin{bmatrix} (a^2 + b^2 + c^2) \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} - 2(ax(t) + by(t) + cz(t)) \begin{bmatrix} a \\ b \\ c \end{bmatrix} \end{bmatrix} \\ &= \frac{1}{a^2 + b^2 + c^2} \begin{bmatrix} -a^2 + b^2 + c^2 & -2ab & -2ac \\ -2ab & a^2 - b^2 + c^2 & -2bc \\ -2ac & -2bc & a^2 + b^2 - c^2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}. \end{aligned} \tag{4}$$

We have proved the following:

**Theorem 3** Let  $S$  be a surface represented by  $\begin{bmatrix} x(s, t) \\ y(s, t) \\ z(s, t) \end{bmatrix}$ . The reflection of  $S$  with respect to the plane

$ax + by + cz = 0$  is given by

$$\begin{bmatrix} p(s, t) \\ q(s, t) \\ r(s, t) \end{bmatrix} = \frac{1}{a^2 + b^2 + c^2} \begin{bmatrix} -a^2 + b^2 + c^2 & -2ab & -2ac \\ -2ab & a^2 - b^2 + c^2 & -2bc \\ -2ac & -2bc & a^2 + b^2 - c^2 \end{bmatrix} \begin{bmatrix} x(s, t) \\ y(s, t) \\ z(s, t) \end{bmatrix}. \tag{5}$$

The same concept also works for finding the reflection of a parametric surface with respect to an arbitrary plane,  $ax + by + cz = d$ . We simply need to make minor adjustment in equation (4) as follows:

**Theorem 4** Let  $S$  be a surface represented by  $\begin{bmatrix} x(s, t) \\ y(s, t) \\ z(s, t) \end{bmatrix}$  and  $(0, 0, \frac{d}{c})$  be the  $z$ -intercept of the plane

$P : ax + by + cz = d$ . Then the reflection of  $S$  with respect to  $P$  is given by

$$\begin{aligned} \begin{bmatrix} p(s, t) \\ q(s, t) \\ r(s, t) \end{bmatrix} & \\ &= \frac{1}{a^2 + b^2 + c^2} \begin{bmatrix} -a^2 + b^2 + c^2 & -2ab & -2ac \\ -2ab & a^2 - b^2 + c^2 & -2bc \\ -2ac & -2bc & a^2 + b^2 - c^2 \end{bmatrix} \begin{bmatrix} x(s, t) - 0 \\ y(s, t) - 0 \\ z(s, t) - \frac{d}{c} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{d}{c} \end{bmatrix} \end{aligned} \tag{6}$$

### 1.3 General Inverses in 2-D

We demonstrate that the 3-D formula (6) does reduce to (1) in 2-D.

When  $c = 0$ , a plane equation  $ax + by + cz + d = 0$  becomes a line of the form  $ax + by + d = 0$ . The slope of this line is  $\frac{-a}{b}$ . Therefore, we set  $\theta = \tan^{-1}(\frac{-a}{b})$  and we have

$$\begin{aligned} \cos^2 \theta &= \frac{1}{\tan^2 \theta + 1} = \frac{b^2}{a^2 + b^2}, \\ \cos 2\theta &= 2 \cos^2 \theta - 1 = \frac{b^2 - a^2}{a^2 + b^2}, \text{ and} \\ \sin 2\theta &= 2 \tan \theta \cos^2 \theta = \frac{-2ab}{a^2 + b^2}. \end{aligned} \tag{7}$$

By using the  $y$ -intercept  $(0, \frac{-d}{b})$  of the line  $ax + by + d = 0$  and (6), we obtain the **general inverse**  $[p(t), q(t)]$  of  $[x(t), y(t)]$  with respect to the line  $ax + by + d = 0$  (i.e.,  $y = \frac{-a}{b}x + \frac{-d}{b}$ ):

$$\begin{aligned} \begin{bmatrix} p(t) \\ q(t) \end{bmatrix} &= \frac{1}{a^2 + b^2} \begin{bmatrix} -a^2 + b^2 & -2ab \\ -2ab & a^2 - b^2 \end{bmatrix} \begin{bmatrix} x(t) - 0 \\ y(t) - (\frac{-d}{b}) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-d}{b} \end{bmatrix} \\ &= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x(t) - 0 \\ y(t) - (\frac{-d}{b}) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-d}{b} \end{bmatrix}. \end{aligned} \tag{8}$$

which coincides with the reflection of  $[x(t), y(t)]$  with respect to a line  $y = mx + b$  in equation (1).

### 1.4 General Inverse Examples in 3D

We apply (6) on Steiner's Roman Surface of the form  $\begin{bmatrix} x(s, t) \\ y(s, t) \\ z(s, t) \end{bmatrix} = \begin{bmatrix} \frac{r^2 \cos^2 t \cdot \sin 2s}{2} \\ \frac{r^2 \sin s \cdot \sin 2t}{2} \\ \frac{r^2 \cos s \cdot \sin 2t}{2} \end{bmatrix}$  with  $s \in [-\pi, \pi]$

and  $t \in [0, 2\pi]$  to find the reflection of  $\begin{bmatrix} x(s, t) \\ y(s, t) \\ z(s, t) \end{bmatrix}$  with respect to the plane  $ax + by + cz = 0$ . The result is shown as follows:

$$\begin{bmatrix} \frac{\frac{1}{2}(-a^2 + b^2 + c^2) \cdot r^2 \cdot \cos^2 t \cdot \sin 2s - abr^2 \cdot \sin s \cdot \sin 2t - acr^2 \cos s \cdot \sin 2t}{a^2 + b^2 + c^2} \\ \frac{-abr^2(\cos^2 t \sin 2s) + \frac{1}{2}(a^2 - b^2 + c^2)r^2(\sin s \cdot \sin 2t) - bcr^2 \cos s \cdot \sin 2t}{a^2 + b^2 + c^2} \\ \frac{-acr^2(\cos^2 t \sin 2s) + \frac{1}{2}(a^2 - b^2 + c^2)r^2(\cos s \cdot \sin 2t) - bcr^2 \sin s \cdot \sin 2t}{a^2 + b^2 + c^2} \end{bmatrix}. \tag{9}$$

We use the following example to demonstrate

**Example 5** Let the surface  $S$  be the Roman Surface represented by  $\begin{bmatrix} \frac{2^2 \cos^2 t \cdot \sin 2s}{2} \\ \frac{2^2 \sin s \cdot \sin 2t}{2} \\ 3 + \frac{2^2 \cos s \cdot \sin 2t}{2} \end{bmatrix}$ . Then the

reflection of  $S$  with respect to the plane  $x + y + 2z = 1$  is

$$\left[ \begin{array}{c} \frac{4 \cos^2 t \cdot \sin 2s - 2 \sin s \cdot \sin 2t - 6 - 4 \cos s \cdot \sin 2t}{3} \\ \frac{-2 \cos^2 t \cdot \sin 2s + 4 \sin s \cdot \sin 2t - 6 - 4 \cos s \cdot \sin 2t}{3} \\ \frac{-4 \cos^2 t \cdot \sin 2s + 4 \sin s \cdot \sin 2t - 3 - 2 \cos s \cdot \sin 2t}{3} + \frac{1}{2} \end{array} \right].$$

We demonstrate the surfaces of  $S$ , its reflection, and  $x + y + 2z = 1$  as in Figure 4. Please see [10] for detailed computations.

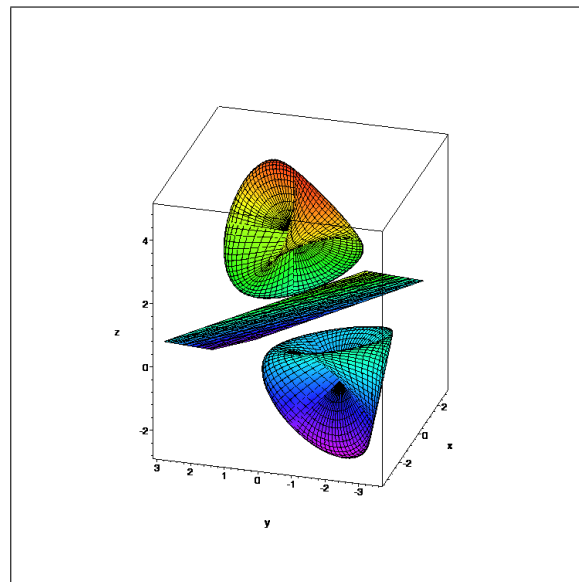


Figure 4. A reflection of Roman Surface with respect to  $x + y + 2z = 1$ .

## 2 Generalizations in 2-D, orthotomic and caustic curves

It is reasonable to discuss the formula for the reflection of a curve  $C_1 = [x_1(t), y_1(t)]$  with respect to a parametric curve  $C_2 = [x(s), y(s)]$ , which we denote it by  $[p(s, t), q(s, t)]$ . Fixing a point on  $C_1$  (i.e., fixing the value of  $t$ ), we want to find the locus  $[p(s, t), q(s, t)]$  of the reflections of  $[x_1(t), y_1(t)]$  with respect to  $C_2$  (i.e., with respect to all tangents of  $C_2$ ).

We recall the general inverse  $[p(t), q(t)]$  of  $[x(t), y(t)]$  with respect to a line  $ax + by + d = 0$  (i.e.,  $y = \frac{-a}{b}x + \frac{-d}{b}$ ) to be

$$\begin{bmatrix} p(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x_1(t) - 0 \\ y_1(t) - \left(\frac{-d}{b}\right) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-d}{b} \end{bmatrix}, \quad (10)$$

where  $\theta = \tan^{-1} \frac{-a}{b} = \tan^{-1} m$ .

Let  $(x(s_0), y(s_0))$  be a point on  $C_2$ . Since the slope  $m$  of the tangent line to  $C_2$  at  $(x(s_0), y(s_0))$  can be expressed by  $\frac{dy}{dx}|_{x=s_0}$ . The tangent line equation to  $C_2$  at  $(x(s_0), y(s_0))$  is

$$\begin{aligned} y &= \left( \frac{dy}{dx} \Big|_{s=s_0} \right) x + \left[ y(s_0) - \left( \frac{dy}{dx} \Big|_{s=s_0} \right) x(s_0) \right]. \\ &= \left[ \left( \frac{\frac{dy}{ds}}{\frac{dx}{ds}} \right) \Big|_{s=s_0} \right] x + \left[ y(s_0) - \left( \frac{\frac{dy}{ds}}{\frac{dx}{ds}} \right) \Big|_{s=s_0} \cdot x(s_0) \right]. \end{aligned} \quad (11)$$

To describe the locus for  $\begin{bmatrix} p(s, t) \\ q(s, t) \end{bmatrix}$  as  $t$  is fixed, we think of  $s_0$  as the variable  $s$ . By (10), we can write

$$\begin{bmatrix} p(s, t) \\ q(s, t) \end{bmatrix} = \begin{bmatrix} \cos \left[ 2 \tan^{-1} \left( \left( \frac{dy}{dx} \right) \right) \right] & \sin \left[ 2 \tan^{-1} \left( \left( \frac{dy}{dx} \right) \right) \right] \\ \sin \left[ 2 \tan^{-1} \left( \left( \frac{dy}{dx} \right) \right) \right] & - \cos \left[ 2 \tan^{-1} \left( \left( \frac{dy}{dx} \right) \right) \right] \end{bmatrix} \cdot \begin{bmatrix} x_1(t) - 0 \\ y_1(t) - \left[ y(s) - \left( \frac{dy}{dx} \right) \cdot x(s) \right] \end{bmatrix} + \begin{bmatrix} 0 \\ \left[ y(s) - \left( \frac{dy}{dx} \right) \Big|_{s=s} \cdot x(s) \right] \end{bmatrix}. \tag{12}$$

For the rest of the paper, a curve  $C_3$  that is the reflection of a curve  $C_1$  with respect to a curve  $C_2$  at a point  $P \in C_2$  is equivalent to a curve  $C_3$  that is the reflection of a curve  $C_1$  with respect to the tangent line to  $C_2$  at a point  $P \in C_2$ . In summary, we have the following

**Theorem 6** Let  $C_1 = [x_1(t), y_1(t)]$  and  $C_2 = [x(s), y(s)]$  be two smooth curves. If  $C_3 = [p(s, t), q(s, t)]$  is the reflection of  $C_1$  with respect to  $C_2$ . Then

$$\begin{bmatrix} p(s, t) \\ q(s, t) \end{bmatrix} = \begin{bmatrix} \cos \left[ 2 \tan^{-1} \left( \left( \frac{dy}{dx} \right) \right) \right] & \sin \left[ 2 \tan^{-1} \left( \left( \frac{dy}{dx} \right) \right) \right] \\ \sin \left[ 2 \tan^{-1} \left( \left( \frac{dy}{dx} \right) \right) \right] & - \cos \left[ 2 \tan^{-1} \left( \left( \frac{dy}{dx} \right) \right) \right] \end{bmatrix} \cdot \begin{bmatrix} x_1(t) - 0 \\ y_1(t) - \left[ y(s) - \left( \frac{dy}{dx} \right) \cdot x(s) \right] \end{bmatrix} + \begin{bmatrix} 0 \\ \left[ y(s) - \left( \frac{dy}{dx} \right) \cdot x(s) \right] \end{bmatrix}. \tag{13}$$

## 2.1 Orthotomic curves

We have geared our discussion into the field of optics. An **orthotomic** curve is the set of reflections of a given point  $O$  with respect to all the tangents of a given curve not passing through  $O$ . In other words, assume we are given a light source  $O$  at a point on  $C_1 = [x_1(t), y_1(t)]$ . Then the locus of the reflections of  $O$  about  $C_2$  is the orthotomic curve of  $C_2$  relative to  $O$ . We mention the following observations and omit their proofs. Instead, we use examples and technological tools to explore these observations and to demonstrate how evolving technological tools allow us to quickly expand our learning horizons to applications.

**Theorem 7** Let  $C_1 = [x_1(t), y_1(t)]$  and  $C_2 = [x(s), y(s)]$  be two smooth curves, and  $C_3 = [p(s, t), q(s, t)]$  is the reflection of  $C_1$  with respect to  $C_2$ . Then the mapping of  $(x_1(t), y_1(t)) \rightarrow \begin{bmatrix} p(s, t) \\ q(s, t) \end{bmatrix}$  is continuous on  $C_1$  (The proof can be obtained from its 3-D case in the Appendix section.)

We mention the following observations and omit their proofs. Instead, we use two examples and technological tools to explore these observations and quickly expand our learning horizons to applications.

**Theorem 8** Let  $[x(s), y(s)]$  be a regular parametric plane curve of class  $C^3$ , and  $P$  be a point which is not a point of  $[x(s), y(s)]$ , and the tangents of  $[x(s), y(s)]$  do not pass through  $P$ . Then the orthotomic curve of  $[x(s), y(s)]$  relative  $P$  has a cusp at  $s = s_0$ , if and only if  $[x(s), y(s)]$  has an inflection point at  $s = s_0$ .

We also make the following observations:

**Remark 9** Let the point  $P \in C_1 \cap C_2$ . Then the reflection of  $P$  with respect to  $C_2$  at the point  $P$  is  $P$  itself.

**Remark 10** Let  $P$  be a point lie on both  $C_1$  and the tangent line to  $C_2$  at a point  $Q$ . Then the reflection of  $P$  with respect to the curve  $C_2$  at the point  $Q$  is  $P$  itself.

**Remark 11** If a curve  $C_1$  is symmetric to  $C_2$  with respect to a tangent line to  $C_2$  at a point  $Q$ , then the reflection of  $C_1$  with respect to  $C_2$  at the point  $Q$  is  $C_2$ .

**Example 12** We refer to Figure 5 below, where  $C_1 = [x_1(t), y_1(t)] = [2 \cos t - \cos 2t, 2 \sin t - \sin 2t]$ ,  $t \in [0, 2\pi]$  (shown in blue cardioid on the left), and  $C_2 = [s, f(s)] = [s, (s-2)^2(s-1)(s+1)]$  (shown in green). We set  $C_3 = [p(t), q(t)]$  to be the reflection of  $C_1$  with respect to the tangent line to  $C_2$  at a point  $A$  (i.e. at a fixed  $s$ ). Then:

(1) For a fixed point  $C$  on  $C_1$  (light source at a point on  $C_1$ ), the orthotomic curve of  $C_2$  relative to  $C$  is shown in black, which can be experimented by using [Geometry Expression] (See [11]) and verified by using [Maple] (See [12]).

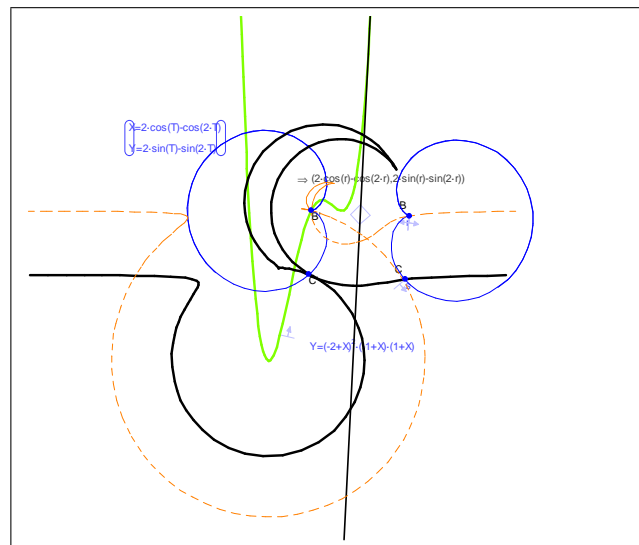


Figure 5. Orthotomic Curve

(2) Picking another point  $B$  on  $C_1$ , we obtain another orthotomic curve (shown in orange in Figure 5). We observe the following behavior, the proof can be found in Appendix.

As  $B$  approaches  $C$ , the orange orthotomic curve approaches the black orthotomic curve.



(3) The sharp corner (cusp) of the black orthotomic occurs at the inflection point of  $C_2$ . This can be experimented with [Geometry Expression]. Detailed proof can be found in [4].

**Example 13** Let  $C_1 = [x_1(t), y_1(t)] = [2 \cos t - \cos 2t, 2 \sin t - \sin 2t]$ ,  $t \in [0, 2\pi]$ , and  $C_2 = [s, f(s)] = [s, (\sin s) - 2]$ . We set  $C_3 = [p(s, t), q(s, t)]$  to be the reflection of  $C_1$  with respect to  $C_2$ . See the graph below.

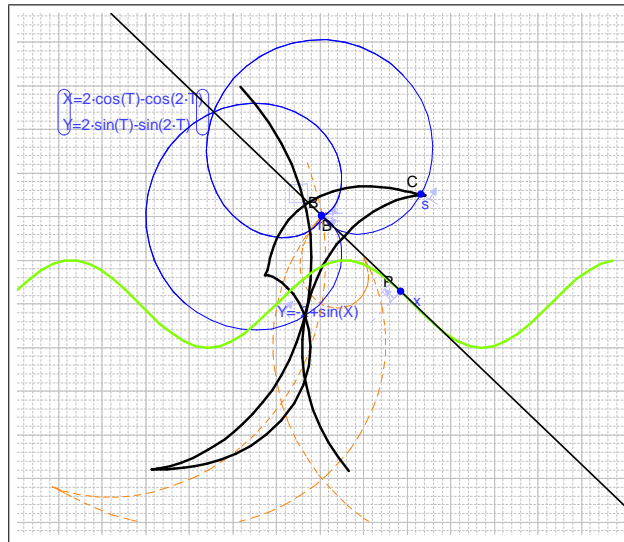


Figure 6. Another Orthotomic Curve.

Then

(1) for a fixed point  $C$  on  $C_1$  (light source at a point on  $C_1$ ), the orthotomic curve of  $C_2$  relative to  $C$  is shown in black in Figure 6. Experimentation with this idea is done by using [Geometry Expression] (see [13]) and verified by using [Maple] (see [14]). We describe here how we find the orthotomic curve of  $C_2$  when we fix a light source at a point on  $C_1$ . We let  $[x_0, y_0] = [s_0, (\sin s_0) - 2]$  be a point on  $C_2$ , then the tangent line equation for  $C_2$  at each  $(x_0, y_0)$  is:

$$y = (\cos s_0) x + (\sin s_0) - 2 - (\cos s_0) s_0.$$

Now we will think of  $s_0$  as a variable  $s$  and notice that the slope is  $\cos s$  so

$$\theta = \tan^{-1}(f'(s)) = \tan^{-1}(\cos(s))$$

and the  $y$  - intercept is

$$[f(s) - f'(s) \cdot s] = (\sin s) - 2 - (\cos s) s.$$

We can write

$$\begin{aligned} \begin{bmatrix} p(s, t) \\ q(s, t) \end{bmatrix} &= \begin{bmatrix} \cos [2 \tan^{-1}(f'(s))] & \sin [2 \tan^{-1}(f'(s))] \\ \sin [2 \tan^{-1}(f'(s))] & -\cos [2 \tan^{-1}(f'(s))] \end{bmatrix} \begin{bmatrix} x_1(t) - 0 \\ y_1(t) - [f(s) - f'(s) \cdot s] \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ [f(s) - f'(s) \cdot s] \end{bmatrix}. \end{aligned} \tag{14}$$

or

$$\begin{aligned} \begin{bmatrix} p(s, t) \\ q(s, t) \end{bmatrix} &= \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} x_1(t) - 0 \\ y_1(t) - ((\sin s) - 2 - (\cos s) s) \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ (\sin s) - 2 - (\cos s) s \end{bmatrix}. \end{aligned} \quad (15)$$

(2) Picking another light source  $B$  on  $C_1$ , we obtain another orthotomic curve (shown in orange in Figure 6). We observe the following behavior, the proof can be found in Appendix.

As  $B \rightarrow C$ , the orange orthotomic curve  $\rightarrow$  the black orthotomic curve.

(3) The sharp corner (cusp) of the black orthotomic occurs at the inflection point of  $C_2$ . We can use [Geometry Expression] to experiment with this result. Detailed proof can be found in [4].

## 2.2 Caustic curves

We now describe how orthotomic curves are used in physics and explore how technological tools accelerate the learning process. In differential geometry, a caustic is the **envelope** of rays either reflected or refracted by a manifold. It is related to the optical concept of caustics. **The evolute of a curve  $C$  is the set of all its centers of curvature. It is equivalent to the envelope of all the normals to  $C$ .** The evolute of a parametrically defined curve  $[x(t), y(t)]$  (see <http://en.wikipedia.org/wiki/Evolute>) is described by:

$$\begin{aligned} X(x, y) &= x - y' \frac{(x')^2 + (y')^2}{x'y'' - x''y'}, \\ Y(x, y) &= y + x' \frac{(x')^2 + (y')^2}{x'y'' - x''y'}. \end{aligned} \quad (16)$$

Alternatively, given an curve  $\mathbf{r}(t)$ , its evolute is described by

$$\mathbf{e}(t) = \mathbf{r}(t) + \frac{\mathbf{n}(t)}{k(t)}, \quad (17)$$

where  $\mathbf{n}(t)$  is the normal vector and  $k(t)$  is the curvature of the curve at  $\mathbf{r}(t)$ , respectively.

To find the caustic generated by rays reflected by a curve  $C$  from a light source  $O$  (caustic of  $C$  relative to  $O$ ) is equivalent to finding the evolute of the orthotomic of  $C$  relative to  $O$ . We explore the caustic curve by using the evolute of the orthotomic, applying equation [16] and with the help of [Maple] in the following examples.

**Example 14** We choose a light source  $O$  on a circle  $C_1 = [2 + 2 \cos t, 2 + 2 \sin t]$  and given an ellipse  $C_2 = [2 \cos s, 3 \sin s]$ . The orthotomic curve  $[p(s, t), q(s, t)]$  of  $C_2$  relative to  $O$  can be found by (13):

$$\begin{aligned} p(s, t) &= \cos \left( 2 \tan^{-1} \left( \frac{3 \cos s}{2 \sin s} \right) \right) (2 + 2 \cos t) - \sin \left( 2 \tan^{-1} \left( \frac{3 \cos s}{2 \sin s} \right) \right) \\ &\quad \left( 2 + 2 \sin t - 3 \sin s - \frac{3 \cos^2 s}{\sin s} \right), \end{aligned} \quad (18)$$

and

$$q(s, t) = -\sin\left(2 \tan^{-1}\left(\frac{3 \cos s}{2 \sin s}\right)\right) (2 + 2 \cos t) - \cos\left(2 \tan^{-1}\left(\frac{3 \cos s}{2 \sin s}\right)\right) \cdot \left(2 + 2 \sin t - 3 \sin s - \frac{3 \cos^2 s}{\sin s}\right) + 3 \sin s + \frac{3 \cos^2 s}{\sin s}. \quad (19)$$

When we fix the light source at  $t = 0$  on  $C_1$ , the orthotomic curve  $[p(s, 0), q(s, 0)]$ , where  $s \in [0, 2\pi]$ , is shown Figure 7, which is done by [Maple] (see [15]).

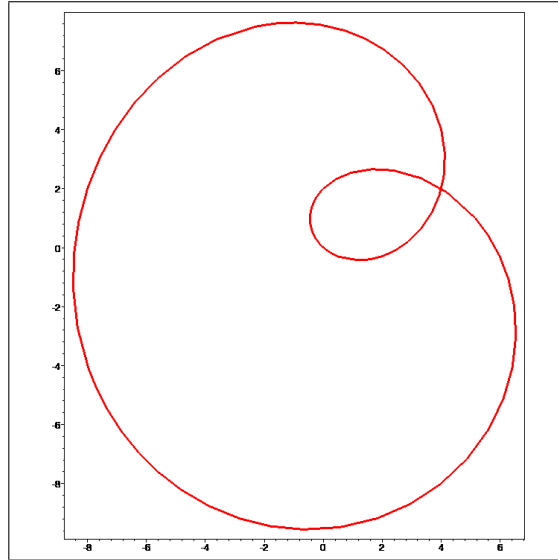


Figure 7. An orthotomic curve when light source is at  $(0,0)$ .

The evolute of  $[p(s, 0), q(s, 0)]$ , or equivalently the caustic of  $C_2$  relative to the light source at  $t = 0$  on  $C_1$ , can be computed by [Maple] (see [15]), which we invite readers to explore. We show the orthotomic curve and its corresponding caustic curve in Figure 8.

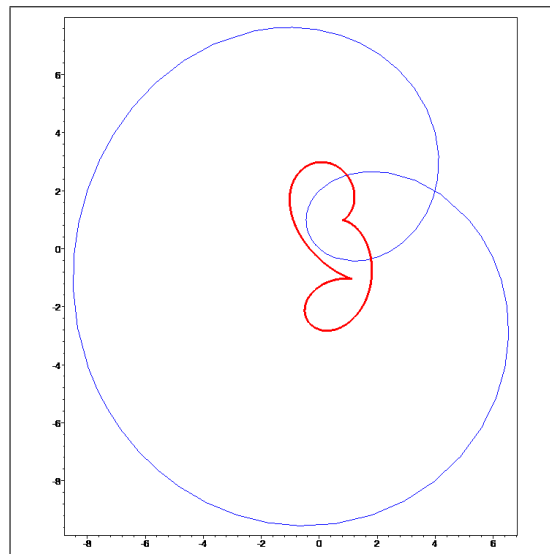


Figure 8. An orthotomic curve and the caustic.

### 3 General Inverses in 3-D, orthotomic and caustic surfaces

We turn our attention to finding the reflection of a light source on a parametric surface  $S_1$  with respect to a parametric surface  $S_2$ . A parametric surface  $\mathbf{w}(u, v)$  is called *smooth* if the tangent vectors in the  $u$  and  $v$  directions satisfy

$$\mathbf{w}_u \times \mathbf{w}_v \neq (0, 0, 0) \text{ throughout its domain.}$$

Given a smooth surface  $\mathbf{w}(u, v) = [x(u, v), y(u, v), z(u, v)]$ . The tangent plane to  $\mathbf{w}$  at the point  $X_0 = (x(u_0, v_0), y(u_0, v_0), z(u_0, v_0))$  on  $\mathbf{w}$  is the plane that contains the tangent vectors  $\mathbf{w}_u, \mathbf{w}_v$  and the point  $X_0$ , and therefore  $\mathbf{w}_u \times \mathbf{w}_v$  is a normal vector of the tangent plane. We will derive the formula for the reflection of a surface  $S_1 = [x_1(s, t), y_1(s, t), z_1(s, t)]$  with respect to the tangent plane to another smooth surface  $S_2 = [x_2(u, v), y_2(u, v), z_2(u, v)]$  at a point  $X_0$  on  $S_2$ . We denote  $\mathbf{w}_u \times \mathbf{w}_v$  at the point  $X_0$  by  $(a(u_0, v_0), b(u_0, v_0), c(u_0, v_0))$ . For brevity, we use  $a = a(u_0, v_0), b = b(u_0, v_0), c = c(u_0, v_0)$  without causing any confusion.

$$\text{Let } X_0 = \begin{bmatrix} x(u_0, v_0) \\ y(u_0, v_0) \\ z(u_0, v_0) \end{bmatrix} \in S_2, X = \begin{bmatrix} x_1(s_0, t_0) \\ y_1(s_0, t_0) \\ z_1(s_0, t_0) \end{bmatrix} \in S_1, \text{ and} \tag{20}$$

$$A = A(u_0, v_0) = \frac{1}{a^2 + b^2 + c^2} \begin{bmatrix} -a^2 + b^2 + c^2 & -2ab & -2ac \\ -2ab & a^2 - b^2 + c^2 & -2bc \\ -2ac & -2bc & a^2 + b^2 - c^2 \end{bmatrix}. \tag{21}$$

Thus, the reflection of  $X = \begin{bmatrix} x_1(s_0, t_0) \\ y_1(s_0, t_0) \\ z_1(s_0, t_0) \end{bmatrix}$  with respect to the tangent plane to  $S_2$  at  $X_0 \in S_2$  is

$$\begin{bmatrix} p(s_0, t_0, u_0, v_0) \\ q(s_0, t_0, u_0, v_0) \\ r(s_0, t_0, u_0, v_0) \end{bmatrix} = A(X - X_0^*) + X_0^*, \tag{22}$$

where  $X_0^* = \begin{bmatrix} 0 \\ 0 \\ \frac{X_0 \cdot (a, b, c)}{c} \end{bmatrix} \in S_2$ .

If we consider  $(u_0, v_0)$  as variables, (22) also provides us the orthotomic surface of  $S_2$  relative to  $X$ .

**Example 15** We pick a light source at a point on the sphere  $S_1 = \begin{bmatrix} x_1(s, t) = 4 + \sin t \cos s \\ y_1(s, t) = 2 + \sin s \sin s \\ z_1(s, t) = -1 + \cos t \end{bmatrix}$ , where  $s \in [0, 2\pi]$  and  $t \in [0, \pi]$ . We want to find the *orthotomic surface* of

$$S_2 = \begin{bmatrix} x_2(u, v) = \frac{7}{5} \sin u \cos v \\ y_2(u, v) = \frac{6}{5} \sin u \sin v \\ z_2(u, v) = \cos u \end{bmatrix},$$

$u \in [0, \pi]$  and  $v \in [0, 2\pi]$ , relative to a light source on  $S_1$ . The orthotomic surface  $\begin{bmatrix} p(0, 0, u, v) \\ q(0, 0, u, v) \\ r(0, 0, u, v) \end{bmatrix}$  can be computed by using equation (22). We plot this surface using [Maple] (see [16]) in Figure 9.

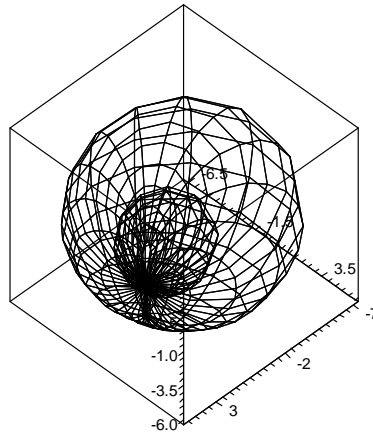


Figure 9. An orthotomic surface.

Naturally, we would like to explore the caustic surface, which is the evolute of an orthotomic surface, or the envelope of an orthotomic surface normals. However, we expect more than one curvature for a curve on a surface in 3-D, and we need more information from differential geometry, which is beyond the scope of this paper. We use the following example to demonstrate that a caustic surface is a natural extension from its caustic curve in 2-D. We use the principle curvatures to obtain the caustic surface (which is called Focal Surface in [2]). First, we use the following example to describe how we obtain the orthotomic surface.

**Example 16** Let the light source  $O$  be at the origin and let  $S$  be the ellipsoid  $[\frac{7}{5} \sin u \cos v, \frac{6}{5} \sin u \sin v, \cos u]$ , where  $u \in [0, \pi]$  and  $v \in [0, 2\pi]$ . The orthotomic surface of  $S$  relative to  $O$  can be computed and plotted by using [Maple] (see [17]). The ellipsoid  $S$  is interior to its orthotomic surface, which are both shown in Figure 10.

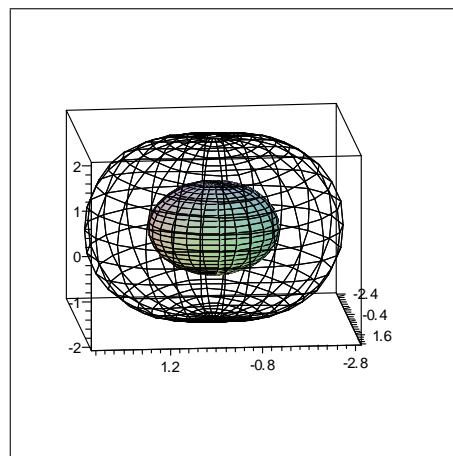


Figure 10. Orthotomic surface for an ellipsoid.

This is understandable since in 2D case, if we consider the light source  $O$  be at  $(0, 0)$ , which is interior to the ellipse  $[\frac{7}{5} \cos s, \frac{6}{5} \sin s]$ , where  $s \in [0, 2\pi]$ , the orthotomic curve of the ellipse relative to  $O$  (see [18]) is

$$\left[ \begin{array}{l} -\sin(2 \arctan(\frac{6 \cos s}{7 \sin s}))(-\frac{6}{5} \sin s - \frac{6 (\cos(s)^2)}{5 \sin s}) \\ -\cos\left(2 \arctan\left(\frac{6 \cos s}{7 \sin s}\right)\right)\left(-\frac{6}{5} \sin s - \frac{6 (\cos s)^2}{5 \sin s}\right) + \frac{6}{5} \sin s + \frac{6 (\cos s)^2}{5 \sin s} \end{array} \right] \quad (23)$$

We plot the ellipse (in green), its orthotomic curve (in blue), and its caustic curve (in red) relative to  $O$  in Figure 11.

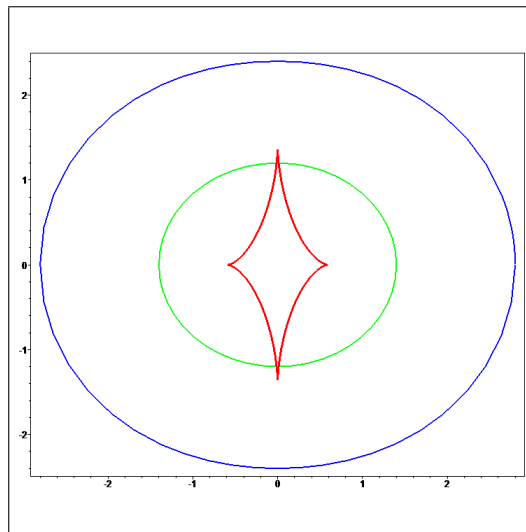


Figure 11. The 2D case for an ellipse, orthotomic, and caustic curve.

Similar to our 2-D discussion, to find the caustic surface, we should continue using the orthotomic surface we have obtained. However, the computations become impossible. We consider the following simplified version.

**Example 17** For simplicity, we assume that the orthotomic surface of a surface  $S$  is the ellipsoid  $[\frac{7}{5} \sin u \cos v, \frac{6}{5} \sin u \sin v, \cos u]$ , where  $u \in [0, \pi]$ , and  $v \in [0, 2\pi]$ . Let the light source  $O$  be at the origin  $(0, 0, 0)$ . The caustic surface of  $S$  relative to  $O$ , when we use the principle curvatures, is shown

in Figure 12 (see [19]):

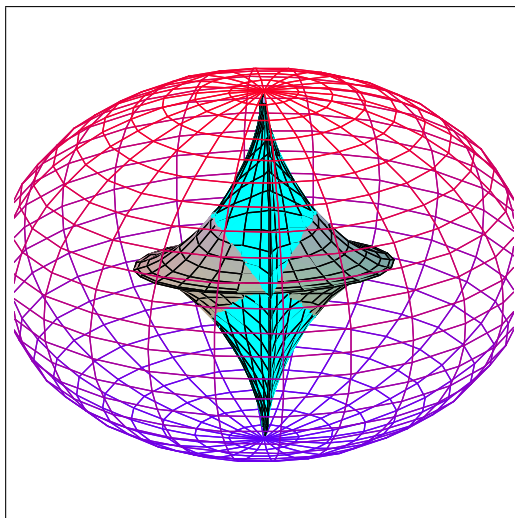


Figure 12. A caustic surface when using principle curvatures.

The plot above makes the 3D caustic surface comprehensible.

## 4 Conclusions

It is evident that geometry software packages, such as ClassPad and Geometry Expressions have allowed us experimentation and conjecture in the process of finding the reflections. We then use Maple, a computer algebra system, to prove our conjectures analytically. We would also like to point out that technological tools prompted us to study more mathematics (in this case in the area of differential geometry) and link mathematical contents to applied areas, in this case, optics of physics (see [1] and [2]) and computer graphics (see [3]). Authors conjecture that evolving technological tools will prompt learners to expand their mathematics knowledge to other applied fields.

## 5 Acknowledgement

The authors would like to thank Phil Todd for many valuable discussions on the use of Geometry Expressions, orthotomic and caustic curves.

## References

- [1] Orthotomic, <http://mathworld.wolfram.com/Orthotomic.html>.
- [2] [http://en.wikipedia.org/wiki/Caustic\\_\(optics\)](http://en.wikipedia.org/wiki/Caustic_(optics))
- [3] Hahmann, S., Blyaeu, A., Buse, L., Elber, G. Mourrain, B., and Ross, C., *Shape Interrogation*, (<http://ljk.imag.fr/membres/Stefanie.Hahmann/PUBLICATIONS/HBBEMR07.pdf>.)

- [4] Hoschek, J., *Smoothing of curves and surfaces*, Computer Aided Geometric Design 2 (1985) pp 97-105, North-Holland. (<http://mrl.nyu.edu/~elif/thesisprop/Hoschek85.pdf>)

## 6 Software Packages

- [5] [ClassPad] A product of CASIO Computer Ltd., <http://classpad.net> or <http://classpad.org/>.
- [6] [Maple] A product of Maplesoft, <http://www.maplesoft.com/>.
- [7] [Geometry Expressions] Geometry Expressions 1.0.55, a product of Saltire Software, 2006, <http://www.geometryexpressions.com/>

## 7 Supplemental Electronic Materials

- [8] Yang, W.-C., *a video clip* which summarizes how ClassPad Manager is used in Section 1.1.
- [9] Yang, W.-C., *ClassPad eActivity* used in Section 1.1.
- [10] Yang, W.-C., Lo, M.-L., *Maple worksheet* for Example 5.
- [11] Todd, P., Yang, W.-C., Lo, M.-L., *Geometry Expressions worksheet* for Example 12.
- [12] Yang, W.-C., Lo, M.-L., *Maple worksheet* for Example 12.
- [13] Todd, P., Yang, W.-C., Lo, M.-L., *Geometry Expressions worksheet* for Example 13.
- [14] Yang, W.-C., Lo, M.-L., *Maple worksheet* for Example 13.
- [15] Yang, W.-C., Lo, M.-L., *Maple worksheet* for Example 14.
- [16] Yang, W.-C., Lo, M.-L., *Maple worksheet* for Example 15.
- [17] Yang, W.-C., Lo, M.-L., *Maple worksheet* for Example 16.
- [18] Yang, W.-C., Lo, M.-L., *another Maple worksheet* for Example 16 .
- [19] Jantzen, R., Nick Korevaar, *Maple worksheet* for Example 18.



## 8 Appendix

Given two surfaces  $S_1 = [x_1(s, t), y_1(s, t), z_1(s, t)]$  and  $S_2 = [x(u, v), y(u, v), z(u, v)]$ , where  $u \in [0, \pi]$ , and  $v \in [0, 2\pi]$ , our goal is to prove that a "small change" in the light source location on  $S_1$  produces a "small change" in the output orthotomic surface of  $S_2$ .

Let  $X = S_1$  and  $Y = \{\text{orthotomic surface of } S_2 \text{ relative to } l : l \in X = S_1\}$

We define a function  $f : X \rightarrow Y$  with  $f(l) := S_l = \text{orthotomic surface of } S_2 \text{ relative to } l \text{ for all } l \in X$ .

Let  $l_0 = [x_1(s_0, t_0), y_1(s_0, t_0), z_1(s_0, t_0)]$  and  $l_1 = [x_1(s_1, t_1), y_1(s_1, t_1), z_1(s_1, t_1)]$  be two points on  $S_1$

and  $(a, b, c) = (a(u, v), b(u, v), c(u, v))$  be a specified normal vector of the tangent plan to  $S_2$  at a point  $[x(u, v), y(u, v), z(u, v)] \in S_2$

Let  $X_0^*(u, v) = \begin{bmatrix} 0 \\ 0 \\ \frac{(a(u,v), b(u,v), c(u,v)) \cdot [x(u,v), y(u,v), z(u,v)]}{c(u,v)} \end{bmatrix}$ , which is the  $z$ - intercept of the tangent plan to  $S_2$  at a point  $[x(u, v), y(u, v), z(u, v)] \in S_2$ .

$$\text{By (22), } f(l) = S_l = \begin{bmatrix} p(s, t, u, v) \\ q(s, t, u, v) \\ r(s, t, u, v) \end{bmatrix} = A(u, v)(l - X_0^*(u, v)) + X_0^*(u, v)$$

In order to consider the closeness of the output orthotomic surfaces, we define the metric (distance function) of the space  $Y$  :

$$d_y(S_{l_0}, S_{l_1}) := \|S_{l_0} - S_{l_1}\| = \sup_{(u,v)} \left\{ \left\| \begin{bmatrix} p(s_0, t_0, u, v) \\ q(s_0, t_0, u, v) \\ r(s_0, t_0, u, v) \end{bmatrix} - \begin{bmatrix} p(s_1, t_1, u, v) \\ q(s_1, t_1, u, v) \\ r(s_1, t_1, u, v) \end{bmatrix} \right\| \right\} \text{ for all } l_0, l_1 \in X.$$

We first prove the following lemma.

**Lemma 18**  $|f(l_0) - f(l_1)| \leq 15 |l_0 - l_1|$  for all  $l_0, l_1 \in X$ .

Proof: We use the same notation for  $l_0$  and  $l_1$  as above.

$$\text{Since } f(l_0) = S_{l_0} = \begin{bmatrix} p(s_0, t_0, u, v) \\ q(s_0, t_0, u, v) \\ r(s_0, t_0, u, v) \end{bmatrix} = A(u, v)(l_0 - X_0^*(u, v)) + X_0^*(u, v)$$

$$|f(l_0) - f(l_1)| = d_y(S_{l_0}, S_{l_1}) := \|S_{l_0} - S_{l_1}\| = \sup_{(u,v)} \{|A(u, v)(l_0 - l_1)|\}$$

$$\begin{aligned} &= \sup_{(u,v)} \left\{ \left\| \frac{1}{a^2+b^2+c^2} \begin{bmatrix} -a^2 + b^2 + c^2 & -2ab & -2ac \\ -2ab & a^2 - b^2 + c^2 & -2bc \\ -2ac & -2bc & a^2 + b^2 - c^2 \end{bmatrix} \begin{bmatrix} x_1(s_0, t_0) - x_1(s_1, t_1) \\ y_1(s_0, t_0) - y_1(s_1, t_1) \\ z_1(s_0, t_0) - z_1(s_1, t_1) \end{bmatrix} \right\| \right\} \\ &= \sup_{(u,v)} \left\{ \left\| \frac{1}{a^2+b^2+c^2} \begin{bmatrix} (-a^2 + b^2 + c^2)(x_1(s_0, t_0) - x_1(s_1, t_1)) - 2ab(y_1(s_0, t_0) - y_1(s_1, t_1)) \\ -2ac(z_1(s_0, t_0) - z_1(s_1, t_1)) - 2ab(x_1(s_0, t_0) - x_1(s_1, t_1)) + \\ (a^2 - b^2 + c^2)(y_1(s_0, t_0) - y_1(s_1, t_1)) - 2bc(z_1(s_0, t_0) - z_1(s_1, t_1)) \\ -2ac(x_1(s_0, t_0) - x_1(s_1, t_1)) - 2bc(y_1(s_0, t_0) - y_1(s_1, t_1)) + \\ (a^2 + b^2 - c^2)(z_1(s_0, t_0) - z_1(s_1, t_1)) \end{bmatrix} \right\| \right\} \\ &\leq \sup_{(u,v)} \left\{ \left\| \frac{1}{a^2+b^2+c^2} [(-a^2 + b^2 + c^2)(x_1(s_0, t_0) - x_1(s_1, t_1)) - 2ab(y_1(s_0, t_0) - y_1(s_1, t_1)) \right. \right. \\ &\quad \left. \left. - 2ac(z_1(s_0, t_0) - z_1(s_1, t_1))] \right\| \right\} \end{aligned}$$

$$\begin{aligned}
 & + \sup_{(u,v)} \left\{ \left| \frac{1}{a^2+b^2+c^2} [-2ab(x_1(s_0, t_0) - x_1(s_1, t_1)) + (a^2 - b^2 + c^2)(y_1(s_0, t_0) - y_1(s_1, t_1)) - 2bc(z_1(s_0, t_0) - z_1(s_1, t_1))] \right| \right\} \\
 & + \sup_{(u,v)} \left\{ \left| \frac{1}{a^2+b^2+c^2} [-2ac(x_1(s_0, t_0) - x_1(s_1, t_1)) - 2bc(y_1(s_0, t_0) - y_1(s_1, t_1)) + (a^2 + b^2 - c^2)(z_1(s_0, t_0) - z_1(s_1, t_1))] \right| \right\} \\
 & \left( \text{Note: since } \left| \frac{-a^2+b^2+c^2}{a^2+b^2+c^2} \right| \leq 1 \text{ and } \left| \frac{-2ab}{a^2+b^2+c^2} \right| \leq \left| \frac{-2ab}{a^2+b^2} \right| \leq 2 \frac{|a|}{\sqrt{a^2+b^2}} \frac{|b|}{\sqrt{a^2+b^2}} \leq 2 \cdot 1 \cdot 1 = 2 \right) \\
 & \leq \{1 \cdot |x_1(s_0, t_0) - x_1(s_1, t_1)| + 2|y_1(s_0, t_0) - y_1(s_1, t_1)| + 2|z_1(s_0, t_0) - z_1(s_1, t_1)|\} \\
 & + \{2|x_1(s_0, t_0) - x_1(s_1, t_1)| + 1 \cdot |y_1(s_0, t_0) - y_1(s_1, t_1)| + 2|z_1(s_0, t_0) - z_1(s_1, t_1)|\} \\
 & + \{2|x_1(s_0, t_0) - x_1(s_1, t_1)| + 2|y_1(s_0, t_0) - y_1(s_1, t_1)| + 1 \cdot |z_1(s_0, t_0) - z_1(s_1, t_1)|\} \\
 & = 5\{|x_1(s_0, t_0) - x_1(s_1, t_1)| + |y_1(s_0, t_0) - y_1(s_1, t_1)| + |z_1(s_0, t_0) - z_1(s_1, t_1)|\} \\
 & \leq 5 \left\{ \left\| \begin{bmatrix} x_1(s_0, t_0) - x_1(s_1, t_1) \\ y_1(s_0, t_0) - y_1(s_1, t_1) \\ z_1(s_0, t_0) - z_1(s_1, t_1) \end{bmatrix} \right\| + \left\| \begin{bmatrix} x_1(s_0, t_0) - x_1(s_1, t_1) \\ y_1(s_0, t_0) - y_1(s_1, t_1) \\ z_1(s_0, t_0) - z_1(s_1, t_1) \end{bmatrix} \right\| + \left\| \begin{bmatrix} x_1(s_0, t_0) - x_1(s_1, t_1) \\ y_1(s_0, t_0) - y_1(s_1, t_1) \\ z_1(s_0, t_0) - z_1(s_1, t_1) \end{bmatrix} \right\| \right\} \\
 & = 15 \left\| \begin{bmatrix} x_1(s_0, t_0) - x_1(s_1, t_1) \\ y_1(s_0, t_0) - y_1(s_1, t_1) \\ z_1(s_0, t_0) - z_1(s_1, t_1) \end{bmatrix} \right\| = 15 |l_0 - l_1|.
 \end{aligned}$$

We now have the main theorem.

**Theorem 19**  $f : X \rightarrow Y$  is continuous on  $X$ .

**Proof:** Let  $l_1 \in X = S_1$ , we must show that  $f$  is continuous at  $l_1$ .

$\forall \varepsilon > 0, \exists \delta = \frac{\varepsilon}{15} > 0$  such that for all  $l_0 \in X = S_1$  and  $|l_0 - l_1| < \delta$

$\implies |f(l_0) - f(l_1)| \leq 15 |l_0 - l_1| < 15 \left(\frac{\varepsilon}{15}\right) = \varepsilon.$